CHROMATIC POLYNOMIALS

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
MASTER OF PHILOSOPHY

By

N. SWAMINATHAN

to the
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

APRIL, 1980

CERTIFICATE

This is to certify that the M. Phil. Thesis entitled 'Chromatic Polynomials' by Mr. N. Swaminathan is a record of bonafide research work carried out by him under my supervision and guidance. He had fulfilled the other requirements for the award of M. Phil. Degree. The results embodied in this thesis have not been submitted elsewhere for a degree or a diploma.

I.I.T. Kanpur
April, 1980

(M.R. Sridharan)
Mathematics Department
I.I.T.Kanpur.

POST GRADUATE OFFICE
This thesis has been approved
for the award of the Degree of
Master of Technology (M.Tech.)
in accordance with the
regulations of the Indian
Institute of Technology Kanpur
Dated. 3.5.81

I.I.T. KANPUR
CENTRAL LIBRARY
Acc. No. A 62321

26 MAY 1988

MATH-1990-M-SWA-CHR

ACKNOWLEDGEMENTS

I take this opportunity to express my sincere thanks to Dr. M.R. Sridharan of the Department of Mathematics, Indian Institute of Technology, Kanpur, India, for his helpful guidance and encouragement throughout the course of this work.

I am extremely grateful to the management of my parent institution, A.V.V.M. Sri Pushpam College, Poondi, Thanjavur District, Tamil Nadu, India, for having granted me the leave for my study under F.I.P. of the U.G.C.

Finally, I thank Mr. G.L. Misra for his excellent typing and Mr. A.N. Upadhyay for careful cyclostyling of the thesis.

N. SWAMINATHAN

CONTENTS

CHAPTER 1

Introduction

In this chapter, we shall describe the origin and development of the study of chromatic polynomials of graphs. Ever since the famous four-color map problem was proposed in 1852 by Francis Gutherie, there have been many attempts to solve the problem including the successful one by Kenneth Appel and Wolfgang Haken in 1976. The solution of the problem apart, the many attempts have given rise to progress in various branches of graph theory and in particular to the study of chromatic polynomials. The early investigations of these polynomials were carried out by Birkhoff [8], [9], mainly in the context of planar graphs. In fact, Birkhoff's work on chromatic polynomials could be viewed as a quantitative approach to the four-color conjecture. Whitney began his researches by extending the quantitative approach so that it could be applied to graphs rather than maps. He [33] developed a method for computing the chromatic polynomial of a graph, wherein he used a general principle or 'logical expansion' which is sometimes called the 'principle of inclusion and exclusion'. Later researchers, however, have tended to use a somewhat different method for the computation of such polynomials. We shall shortly consider both the methods in brief. We begin with a few basic definitions.

1.1 Basic Definitions

Definition 1.1. A graph G is a pair $(V(G),E(G))$ written usually (V,E), where V is a finite non-empty set of elements called vertices and E is a finite set of unordered pairs of distinct elements of V called edges. V is called the vertex set and E, the edge-set of G. A graph can be visualised simply by letting each vertex be a point in the plane and letting each edge be a line segment or curve joining the two vertices involved. A graph with p vertices and q edges is called a (p,q) graph. The $(1,0)$ graph is trivial.

Definition 1.2. The null graph N_p is the graph with p vertices and no edges. Sometimes it is useful to introduce the empty graph, which consists of no vertices or edges. A graph in which every two vertices are adjacent is called a complete graph and is denoted by K_p.

Definition 1.3. If $e = \{u,v\}$, henceforth denoted $e = uv$, is an edge of G then u and v are said to be adjacent. Also u and e are said to be incident as are v and e. Furthermore, two edges of G incident to the same vertex are called adjacent edges.

Definition 1.4. Two graphs G and H are said to be isomorphic (written $G \cong H$) if there is a one-to-one correspondence between their vertex sets which preserves the adjacency of vertices.

Definition 1.5. For each vertex v in a graph G, the number of edges incident to v is called the valency of v, denoted by

$\rho(v)$. A vertex of valency 0 is called an isolated vertex and a vertex of valency 1 is called an end-vertex.

Definition 1.6. A subgraph of G is a graph having all of its vertices and edges in G. A spanning subgraph is a subgraph containing all the vertices of G. If e is an edge of G, then the edge-deleted subgraph G-e is the graph obtained from G by removing the edge e. Similarly, if v is a vertex of G, then the vertex deleted subgraph G-v is the graph obtained from G by removing the vertex v together with all the edges incident to v.

Definition 1.7. If e = vw is an edge of G, a new graph from G can be obtained by removing the edge e and identifying v and w in such a way that the resulting vertex is incident to all those edges (other than e) which were originally incident to v or w; this is called contracting the edge e.

Definition 1.8. If G and G' are disjoint graphs, then their union $G \cup G'$ is the graph with vertex-set $V(G) \cup V(G')$ and edge-set $E(G) \cup E(G')$. The join of G and G', denoted by G + G' is obtained from their disjoint union by adding edges joining each vertex of G to each vertex of G'. The wheel W_p is the graph $C_{p-1} + K_1$.

Definition 1.9. If G is a graph, the complement of G (denoted by $\bar{G}$) is the graph with the same vertex-set as G, whose vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. A graph is said to be self-complementary if it is isomorphic to its complement.

<u>Definition</u> 1.10. An alternating ~~sequence~~ sequence of vertices and edges starting with a vertex and ending with a vertex say of the form $v_0, v_0v_1, v_1, v_1v_2, v_2, \ldots, v_{r-1}, v_{r-1}v_r, v_r$ is called a walk of length r from v_0 to v_r. If the vertices are all distinct, the walk is called a path. A path in which the vertices $v_0, v_1, \ldots, v_r$ are all distinct except for v_0 and v_r (which coincide) is called a circuit denoted by C_r. C_3 is often called a triangle. A path with r vertices is denoted by P_r.

<u>Definition</u> 1.11. A graph G is connected if there is a path joining each pair of vertices of G. A maximal connected subgraph of G is called a component of G. The number of components of G is denoted by k(G).

<u>Definition</u> 1.12. A vertex v of a graph G is called a cut-vertex of G if $k(G-v) > k(G)$. A bridge of a graph G is an edge e such that $k(G-e) > k(G)$.

<u>Definition</u> 1.13. A tree is a connected graph which contains no circuits. If G is a connected graph, then a spanning tree in G is a connected spanning subgraph containing no circuits.

<u>Definition</u> 1.14. A planar graph is a graph which can be embedded in the plane in such a way that no two edges intersect except at a vertex to which they are both incident. A graph embedded in the plane in this way is called a plane graph. When every region of the plane graph is bounded by a triangle, the graph is called a triangulated plane graph.

Definition 1.15. A proper coloring of a graph G is an assignment of colors to its vertices in such a way that no two adjacent vertices have the same color. If a proper coloring of G is to be considered distinct from another proper coloring of G obtained from the first by a permutation of the colors, then this is referred to as a proper coloring with color difference. In our subsequent discussions, whenever the term 'coloring' occurs, it will mean a proper coloring with color difference accounted for. The chromatic number $\chi(G)$ of a graph G is the minimum number of colors needed to color the vertices of G.

Definition 1.16. If λ is the number of colors, we can associate a function denoted by $M_G(\lambda)$ with a given graph G, which expresses the number of different ways of coloring G as a function of the number of specified colors. The function $M_G(\lambda)$, as we are about to see, is a polynomial in λ of degree p. This polynomial is called the chromatic polynomial of G.

1.2. $M_G(\lambda)$ as a polynomial in λ

We are now going to see the method developed by Whitney [33] for obtaining $M_G(\lambda)$. His method makes use of a general principle which we give below.

Let there be a finite set of objects and certain properties $A_1, A_2, \ldots, A_m$. Let n denote the total number

of objects. Let $n(A_1)$ denote the number of objects with the property A_1, $n(\bar{A}_1)$ the number of objects without the property A_1, $n(A_1A_2)$ the number of objects with both properties A_1 and A_2, $n(\bar{A}_1\bar{A}_2)$ the number of objects with neither property and so on. Then

$$\begin{aligned} n(\bar{A}_1\bar{A}_2, \ldots, \bar{A}_m) = n &- [n(A_1) + n(A_2) + \ldots + n(A_m)] \\ &+ [n(A_1A_2) + n(A_1A_3) + \ldots + n(A_{m-1}A_m)] \\ &- [n(A_1A_2A_3) + \ldots] + \ldots \\ &+ (-1)^m\, n(A_1A_2, \ldots, A_m). \end{aligned}$$

The foregoing principle is made use of in obtaining $M_G(\lambda)$. Let there be p vertices in the graph G, say $v_1, v_2, \ldots, v_p$ and let v_1v_2, $v_2v_4, \ldots, v_3v_p$ be the edges of G. There are λ^p possible colorings, formed by giving each vertex in succession any one of the λ colors. Call this set of colorings as R. Let $A_{v_1v_2}$ denote those colorings with the property that v_1 and v_2 are of the same color, $A_{v_2v_4}$ denote those colorings with the property that v_2 and v_4 are of the same color, etc. Then the set of admissible colorings is

$$\bar{A}_{v_1v_2}\ \bar{A}_{v_2v_4}\ , \ldots,\ \bar{A}_{v_3v_p}$$

and if there are q edges in the graph, the number of colorings is

$$M_G(\lambda) = n(\bar{A}_{v_1v_2}\ \bar{A}_{v_2v_4}, \ldots, \bar{A}_{v_3v_p})$$

$$= n - [n(A_{v_1v_2}) + n(A_{v_2v_4}) + \ldots + n(A_{v_3v_p})]$$

$$+ [n(A_{v_1v_2}\ A_{v_2v_4}) + \ldots] - \ldots$$

$$+ (-1)^q\, n(A_{v_1v_2}\ A_{v_2v_4}, \ldots, A_{v_3v_p}).$$

With each property $A_{v_1v_2}$ is associated an edge v_1v_2 of G. In the expansion, there is a term corresponding to every possible combination of the properties $A_{v_iv_j}$; with this combination, we associate the corresponding edges, forming a subgraph H of G. We let H contain all the vertices of G.

Let us evaluate a typical term, such as $n(A_{v_1v_2}\ A_{v_1v_4} \ldots\ldots A_{v_3v_j})$. This is the number of ways of coloring G in λ or fewer colors in such a way that v_1 and v_2 are of the same color, v_1 and v_4 are of the same color, ..., v_3 and v_j are of the same color. In the corresponding subgraph H, any two vertices that are joined by an edge must be of the same color and thus all the vertices in a single connected piece in H are of the same color. If there are k connected pieces in H, the value of this term is λ^k. If there are s edges in H, the sign of the term is $(-1)^s$. Thus

$$(-1)^s\, n(A_{v_1v_2}\, A_{v_1v_4}\, \ldots\, A_{v_3v_j}) = (-1)^s\, \lambda^k.$$

If there are $N_{(k,s)}$ subgraphs of s edges in k connected pieces, the corresponding terms contribute to $M_G(\lambda)$ an amount $(-1)^s\, N_{(k,s)}\, \lambda^k$. Hence summing over all possible values of k and s, we find

$$M_G(\lambda) = \sum_{k,s} (-1)^s\, N_{(k,s)}\, \lambda^k,$$

which is a polynomial in λ.

Of course, the above expression for $M_G(\lambda)$ was further simplified by Whitney [33], by defining what are known as the broken circuits of G.

As we have mentioned in the beginning, there is another method by which $M_G(\lambda)$ could be obtained. To this end, we first observe that if G is the null graph with p vertices, $M_G(\lambda) = \lambda^p$ and if G is the complete graph with p vertices, $M_G(\lambda) = \lambda(\lambda-1)(\lambda-2)\,\ldots\,(\lambda-p+1)$.

We now have a fundamental theorem [25]. The theorem asserts that if G is a graph with p vertices containing an edge e and if G' and G" are the graphs obtained from G by respectively deleting and contracting the edge e, then

$$M_G(\lambda) = M_{G'}(\lambda) - M_{G''}(\lambda).$$

We observe that G' is a graph, whose number of edges is one fewer than G and G'' is a graph with (p-1) vertices. The process of deletion and contraction of an edge can now be applied to G' and G'' separately and so on. If the operation is repeated as often as necessary, we will end up at a stage when all the graphs that have been derived from the original graph are without edges. At this stage, it is clearly impossible to go any further and $M_G(\lambda)$ will be expressed in terms of powers of λ. Since at each stage of reduction, there is exactly one graph having p vertices, it is true at the final stage too. Thus $M_G(\lambda)$ is a monic polynomial in λ of degree p.

1.3. Chromatic polynomials with respect to planar graphs

As we have mentioned in the beginning, the early investigations of chromatic polynomials were mainly in terms of planar graphs. A wealth of information has been obtained concerning the properties of the chromatic polynomials of planar graphs. The early works on chromatic polynomials were mainly motivated by the four-color conjecture. In fact, the four-color conjecture can be reformulated in terms of chromatic polynomial as follows.

Reformulation of the four-color conjecture. If G is a planar graph, $M_G(4) > 0$.

More recently, Berman and Tutte [5] noticed a remarkable property of chromatic polynomials of triangulated

plane graphs. They observed empirically that if G is a triangulated plane graph, then $M_G(\lambda)$ tends to have a root near $\lambda = 1 + \tau$, although $1 + \tau$ itself is not the root of any chromatic polynomial, where $\tau = \frac{1 + \sqrt{5}}{2}$, the positive root of the quadratic equation $x^2 = x + 1$. The number τ is called the classical 'golden ratio', believed to express the ideal ratio of the length of a rectangle to its width. The root $\lambda = 1 + \tau$, for this reason, is called the golden root. An explanation of this interesting phenomenon came a little later when Tutte [30] produced the following theorem.

Theorem. If $M \in Z(k)$, then

$|P(M, 1 + \tau)| \leq \tau^{5-k}$, where $Z(k)$ is the class of all triangulated plane graphs with exactly k vertices and $P(M, \lambda)$ is the chromatic polynomial of M.

Naturally, when k is large enough, $P(M, \lambda)$ tends to have a zero near $1 + \tau$.

1.4. An application. Our aim in this section is to emphasize the fact that the concept of chromatic polynomials has applications to practical situations. We shall now mention one application, which can be found in [25].

Allocation of channels to television stations. Assume that there are k possible channels available for use by the

n television stations in a certain country. It is natural that stations which are near to each other cannot use the same channel without causing interference. Thus, given any two stations, it may or may not be the case that they can use the same channel. The problem is to allocate a channel to each station in such a way that any two stations which need to have different channels get different channels.

This situation can be converted into a graph theoretical problem. Construct a graph G whose vertices in G represent stations. Two vertices in G are joined by an edge if and only if the corresponding stations cannot use the same channel. Then any allocation of channels is, effectively, a coloring of G in k colors and if it is proper then the condition that nearby stations receive different channels is satisfied. Thus, the problem reduces to that of coloring a graph and the chromatic polynomial will give the number of ways of allocating the k channels.

We have thus seen in this chapter how with every graph G, a polynomial $M_G(\lambda)$ known as the chromatic polynomial of G is associated. We have also mentioned some of the works on chromatic polynomials of planar graphs and cited a practical application of the concept of chromatic polynomials. In the next chapter, we shall review some of the familiar properties of chromatic polynomials.

CHAPTER 2

Properties of chromatic polynomials

In this chapter, we shall review some of the familiar properties of chromatic polynomials of graphs. Mention is also made of the chromatic polynomials of certain kinds of graphs.

2.1. Some general properties of chromatic polynomials

Theorem 2.1.1 [25]. $M_G(\lambda)$ has no constant term.

Proof. $M_G(0)$ stands for the number of ways of coloring a graph G with no colors and naturally it must be zero. This establishes that $M_G(\lambda)$ has no constant term.

Theorem 2.1.2 [25] The terms in $M_G(\lambda)$ alternate in sign.

Corollary 2.1.1 [25] The coefficient of λ^{p-1} in $M_G(\lambda)$ is $-q$, where G is a (p,q) graph.

Theorem 2.1.3. [25] If a graph G has components $G_1, G_2, \ldots, G_k$, then

$$M_G(\lambda) = M_{G_1}(\lambda) \cdot M_{G_2}(\lambda) \ldots \cdot M_{G_k}(\lambda).$$

Proof. Since the components are disjoint, the coloring of each is quite independent of the coloring of the others. Hence, the number of ways of coloring the whole graph is

simply the product of the numbers of colorings of the separate components.

Theorem 2.1.4. [25] If two graphs X and Y overlap in a complete graph on k vertices, then the chromatic polynomial of the graph formed by X and Y together is $\frac{M_X(\lambda)\, M_Y(\lambda)}{\lambda^{(k)}}$, where $\lambda^{(k)} = \lambda(\lambda-1)(\lambda-2)\,.....\,(\lambda-k+1)$.

2.2. Chromatic polynomials of certain kinds of graphs

Theorem 2.2.1. [25] A graph G with p vertices is a tree if and only if $M_G(\lambda) = \lambda(\lambda-1)^{p-1}$.

Theorem 2.2.2. [25] $M_{C_n}(\lambda) = (\lambda-1)^n + (-1)^n(\lambda-1)$.

Theorem 2.2.3. [30] $M_{W_{k+1}}(\lambda) = \lambda\left[(\lambda-2)^k+(-1)^k(\lambda-2)\right]$

2.3. Lower bounds for the chromatic coefficients

Theorem 2.3.1. [25]. If a graph G is connected, then the absolute value of the coefficient of λ^r in $M_G(\lambda)$ is not less than $\binom{p-1}{r-1}$, p being the number of vertices of G.

Corollary 2.3.1. [25] The smallest number r such that λ^r has a non-zero coefficient in $M_G(\lambda)$ is the number of components of G.

Theorem 2.3.2 [15] If G is a connected graph on p vertices and q edges and

$$M_G(\lambda) = \sum_{r=1}^{p} (-1)^{p-r}\, a_r\, \lambda^r, \text{ then}$$

$$a_r \geq (q - p+2) \binom{p-2}{r-1} + \binom{p-2}{r-2}.$$

2.4. Some information about chromatic polynomial of a graph

Theorem 2.4.1. [32] If $\lambda < 0$ and if G is a connected graph on p vertices, then $M_G(\lambda)$ is non-zero with the sign of $(-1)^p$.

Theorem 2.4.2. [32] If G is a connected graph on p vertices and if we write

$$M_G(\lambda) = \lambda P_G(\lambda), \quad \text{then for} \quad \lambda < 1,$$

$P_G(\lambda)$ is non-zero with the sign of $(-1)^{p-1}$.

Corollary 2.4.1. [32] In the open interval $0 < \lambda < 1$, $M_G(\lambda)$ is non-zero with the sign of $(-1)^{p-1}$.

We have thus mentioned in this chapter some of the properties of chromatic polynomials, some bounds for the coefficients of chromatic polynomials and also certain results regarding the chromatic polynomial. In the next chapter, we shall present a list of chromatic polynomials of graphs upto 5 vertices and find out the chromatic roots for some graphs. We shall also mention a few unsolved problems.

CHAPTER 3

Chromatic polynomials and chromatic roots

In this chapter, we shall be concerned with some more results on chromatic polynomials and thereby chromatic roots. By a chromatic root we mean the following. If $M_G(\lambda)$ is the chromatic polynomial of a graph G, then a root of the equation $M_G(\lambda) = 0$ is called a chromatic root. We begin by listing chromatic polynomials of graphs upto 5 vertices.

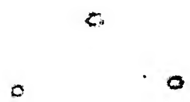
3.1. Chromatic polynomials of graphs upto 5 vertices

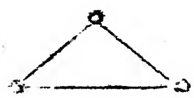
In writing down the chromatic polynomials of graphs upto 5 vertices, we adopt the following method. A graph G and its complement $\bar{G}$ are drawn one below the other and their chromatic polynomials are written against each. Wherever there is no mention of $\bar{G}$, it means that G is self-complementary.

G :  $M_G(\lambda) = \lambda$

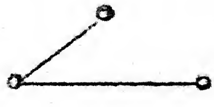
G :  $M_G(\lambda) = \lambda^2$

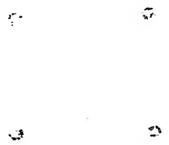
$\overline{G}$:  $M_{\overline{G}}(\lambda) = \lambda^2 - \lambda$

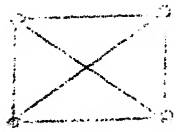
G :  $M_G(\lambda) = \lambda^3$

$\overline{G}$:  $M_{\overline{G}}(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda$

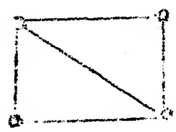
G :  $M_G(\lambda) = \lambda^3 - \lambda^2$

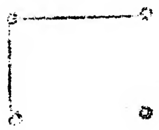
$\overline{G}$:  $M_{\overline{G}}(\lambda) = \lambda^3 - 2\lambda^2 + \lambda$

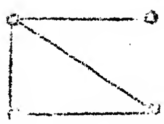
G :  $M_G(\lambda) = \lambda^4$

$\bar{G}$:  $M_{\bar{G}}(\lambda) = \lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda$

G :  $M_G(\lambda) = \lambda^4 - \lambda^3$

$\bar{G}$:  $M_{\bar{G}}(\lambda) = \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda$

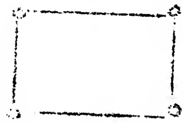
G :  $M_G(\lambda) = \lambda^4 - 2\lambda^3 + \lambda^2$

$\bar{G}$:  $M_{\bar{G}}(\lambda) = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda$

G :

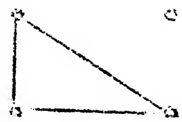
$$M_G(\lambda) = \lambda^4 - 2\lambda^3 + \lambda^2$$

$\bar{G}$:



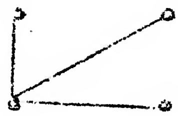
$$M_{\bar{G}}(\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$$

G :



$$M_G(\lambda) = \lambda^4 - 3\lambda^3 + 2\lambda^2$$

$\bar{G}$:



$$M_{\bar{G}}(\lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$$

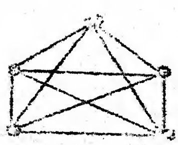
G :

$$M_G(\lambda) = \lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$$

G :

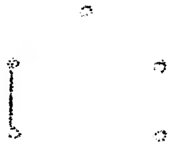
$$M_G(\lambda) = \lambda^5$$

$\bar{G}$:



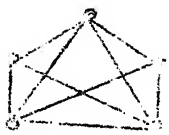
$$M_{\bar{G}}(\lambda) = \lambda^5 - 10\lambda^4 + 35\lambda^3 - 50\lambda^2 +$$

G :



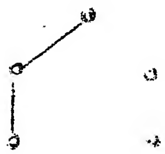
$M_G(\lambda) = \lambda^5 - \lambda^4$

$\bar{G}$:



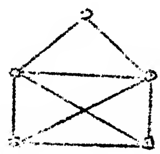
$M_{\bar{G}}(\lambda) = \lambda^5 - 9\lambda^4 + 29\lambda^3 - 39\lambda^2 + 18\lambda$

G :



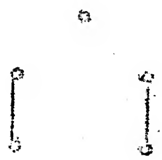
$M_G(\lambda) = \lambda^5 - 2\lambda^4 + \lambda^3$

$\bar{G}$:



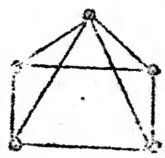
$M_{\bar{G}}(\lambda) = \lambda^5 - 8\lambda^4 + 23\lambda^3 - 28\lambda^2 + 12\lambda$

G :

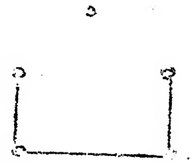


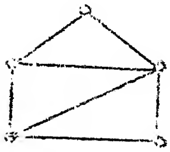
$M_G(\lambda) = \lambda^5 - 2\lambda^4 + \lambda^3$

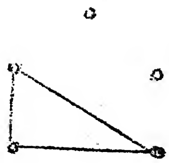
$\bar{G}$:

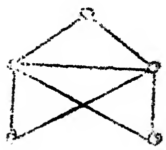


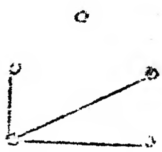
$M_{\bar{G}}(\lambda) = \lambda^5 - 8\lambda^4 + 24\lambda^3 - 31\lambda^2 + 14\lambda$

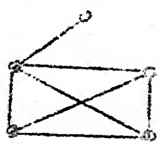
G :  $M_G(\lambda) = \lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2$

$\bar{G}$:  $M_{\bar{G}}(\lambda) = \lambda^5 - 7\lambda^4 + 18\lambda^3 - 20\lambda^2 + 8\lambda$

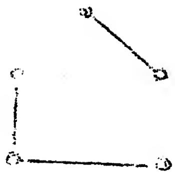
G :  $M_G(\lambda) = \lambda^5 - 3\lambda^4 + 2\lambda^3$

$\bar{G}$:  $M_{\bar{G}}(\lambda) = \lambda^5 - 7\lambda^4 + 18\lambda^3 - 20\lambda^2 + 8\lambda$

G :  $M_G(\lambda) = \lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2$

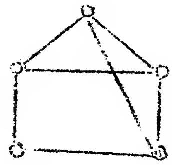
$\bar{G}$:  $M_{\bar{G}}(\lambda) = \lambda^5 - 7\lambda^4 + 17\lambda^3 - 17\lambda^2 + 6\lambda$

G :



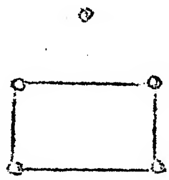
$$M_G(\lambda) = \lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2$$

$\bar{G}$:



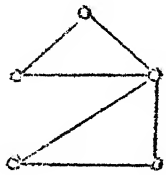
$$M_{\bar{G}}(\lambda) = \lambda^5 - 7\lambda^4 + 19\lambda^3 - 23\lambda^2 + 10\lambda$$

G :



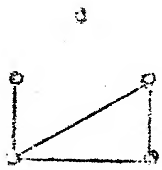
$$M_G(\lambda) = \lambda^5 - 4\lambda^4 + 6\lambda^3 - 3\lambda^2$$

$\bar{G}$:



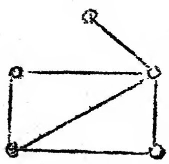
$$M_{\bar{G}}(\lambda) = \lambda^5 - 6\lambda^4 + 13\lambda^3 - 12\lambda^2 + 4\lambda$$

G :



$$M_G(\lambda) = \lambda^5 - 4\lambda^4 + 5\lambda^3 - 2\lambda^2$$

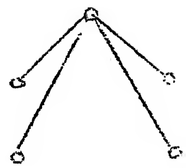
$\bar{G}$:



$$M_{\bar{G}}(\lambda) = \lambda^5 - 6\lambda^4 + 13\lambda^3 - 12\lambda^2 + 4\lambda$$

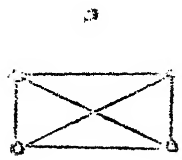
I.I.T. KANPUR
CENTRAL LIBRARY
Acc. No. A 62321

G :



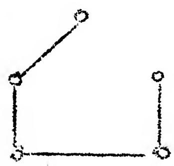
$$M_G(\lambda) = \lambda^5 - 4\lambda^4 + 6\lambda^3 - 4\lambda^2 + \lambda$$

$\bar{G}$:



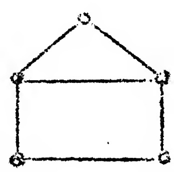
$$M_{\bar{G}}(\lambda) = \lambda^5 - 6\lambda^4 + 11\lambda^3 - 6\lambda^2$$

G :



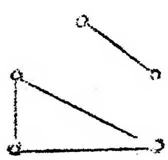
$$M_G(\lambda) = \lambda^5 - 4\lambda^4 + 6\lambda^3 - 4\lambda^2 + \lambda$$

$\bar{G}$:



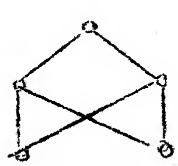
$$M_{\bar{G}}(\lambda) = \lambda^5 - 6\lambda^4 + 14\lambda^3 - 15\lambda^2 + 6\lambda$$

G :



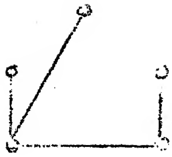
$$M_G(\lambda) = \lambda^5 - 4\lambda^4 + 5\lambda^3 - 2\lambda^2$$

$\bar{G}$:



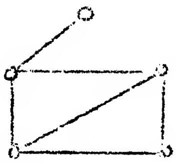
$$M_{\bar{G}}(\lambda) = \lambda^5 - 6\lambda^4 + 15\lambda^3 - 17\lambda^2 + 7$$

G :



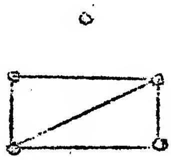
$$M_G(\lambda) = \lambda^5 - 4\lambda^4 + 6\lambda^3 - 4\lambda^2 + \lambda$$

$\bar{G}$:



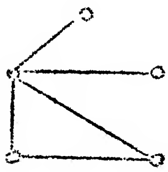
$$M_{\bar{G}}(\lambda) = \lambda^5 - 6\lambda^4 + 13\lambda^3 - 12\lambda^2 + 4\lambda$$

G :



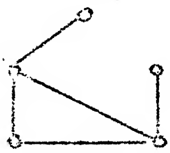
$$M_G(\lambda) = \lambda^5 - 5\lambda^4 + 8\lambda^3 - 4\lambda^2$$

$\bar{G}$:



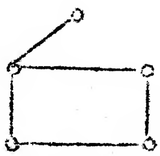
$$M_{\bar{G}}(\lambda) = \lambda^5 - 5\lambda^4 + 9\lambda^3 - 7\lambda^2 + 2\lambda$$

G :



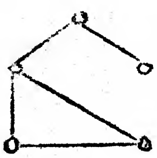
$$M_G(\lambda) = \lambda^5 - 5\lambda^4 + 9\lambda^3 - 7\lambda^2 + 2\lambda$$

G :



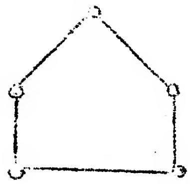
$$M_G(\lambda) = \lambda^5 - 5\lambda^4 + 10\lambda^3 - 9\lambda^2 + 3\lambda$$

$\bar{G}$:



$$M_{\bar{G}}(\lambda) = \lambda^5 - 5\lambda^4 + 9\lambda^3 - 7\lambda^2 + 2\lambda$$

G :



$$M_G(\lambda) = \lambda^5 - 5\lambda^4 + 10\lambda^3 - 10\lambda^2 + 4\lambda$$

3.2. Chromatic roots in some particular cases

Theorem 3.2.1. The chromatic polynomials of graphs with p vertices and 1, 2 or 3 edges have integer roots only.

Proof. The theorem follows as an immediate consequence of the results obtained in section 3.1, provided we add suitably additional isolated vertices to the graphs to make them graphs with p vertices. Once this is done, we obtain the following.

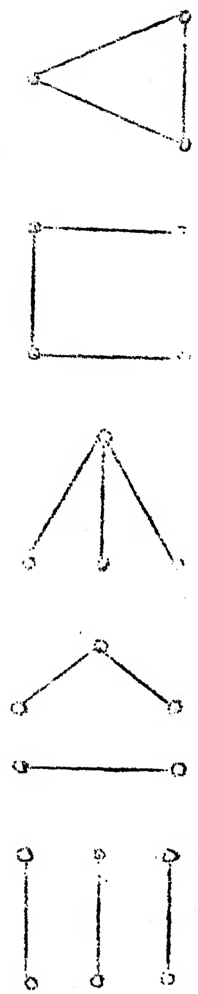
If G is a graph with p vertices and only one edge,

$$M_G(\lambda) = \lambda^p - \lambda^{p-1}.$$

If G is a graph with p vertices and two edges, then the two edges may have a vertex in common or may be disjoint. In either case,

$$M_G(\lambda) = \lambda^p - 2\lambda^{p-1} + \lambda^{p-2}$$

If G is a graph with p vertices and 3 edges, then the 3 edges may assume one of the following five forms, the remaining vertices in each case being isolated vertices.



In the first case, $M_G(\lambda) = \lambda^{p-3}\,\lambda(\lambda-1)(\lambda-2)$

$$= \lambda^p - 3\lambda^{p-1} + 2\lambda^{p-2}$$

In all the remaining cases, $M_G(\lambda) = \lambda^{p-3}(\lambda-1)^3$

$$= \lambda^p - 3\lambda^{p-1} + 3\lambda^{p-2} - \lambda^{p-3}$$

Thus it follows that the chromatic roots of graphs with p vertices and 1, 2 or 3 edges are only integers.

Theorem 3.2.2. Let q be the number of edges in the complete graph K_p. The chromatic polynomials of graphs with p vertices and q or q-1 edges have integer roots only.

Proof. When the graph G has p vertices and q edges,

$$M_G(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+1) \quad \text{and}$$

obviously the chromatic roots are only integers.

Next, suppose G is a graph with p vertices and q-1 edges. We can add an edge, say e, suitably to G so that it becomes the complete graph K_p. The contraction of the edge e in K_p will result in K_{p-1}. Hence, by the fundamental theorem in section 1.2,

$$M_{K_p}(\lambda) = M_G(\lambda) - M_{K_{p-1}}(\lambda)$$

$$\therefore \quad M_G(\lambda) = M_{K_p}(\lambda) + M_{K_{p-1}}(\lambda)$$

$$= \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2)(\lambda-p+1)$$

$$+ \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2)$$

$$= \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2)(\lambda-p+1+1)$$

$$= \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2)^2.$$

Hence, in this case too, the chromatic roots are only integers.

Theorem 3.2.3. Let q be the number of edges in K_p. There are exactly two graphs G_1 and G_2 with p vertices and q-2 edges and their chromatic polynomials are

$$M_{G_1}(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+3)^2 (\lambda-p+2)$$

$$M_{G_2}(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+3)[\lambda^2-\lambda(2p-5)+p^2-5p+7]$$

Proof. Consider any graph with p vertices and q-2 edges. If two edges are added suitably to the graph, it becomes K_p. Now, there are two possibilities.

(i) The two missing edges in the graph have a vertex in common. Call such a graph G_1.

(ii) The two missing edges in the graph are disjoint. Call such a graph G_2.

Now, consider G_1. By adding an edge to G_1, we get a graph G_1' with p vertices and (q-1) edges and on contraction of this edge in G_1' we get the complete graph K_{p-1}. Hence, by the fundamental theorem in section 1.2,

$$M_{G_1'}(\lambda) = M_{G_1}(\lambda) - M_{K_{p-1}}(\lambda)$$

$$\therefore \quad M_{G_1}(\lambda) = M_{G_1'}(\lambda) + M_{K_{p-1}}(\lambda)$$

$$= \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2)^2$$

$$+ \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2)$$

$$= \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2) \left[\lambda-p+2+1 \right]$$

$$= \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+3)^2 (\lambda-p+2)$$

Next, consider G_2. By adding an edge to G_2, we get a graph G_2' with p vertices and (q-1) edges and on contraction of this edge in G_2', we obtain a graph G_2'' with (p-1) vertices and whose number of edges is one less than that in the complete graph K_{p-1}. Hence, by the fundamental theorem in section 1.2,

$$M_{G_2'}(\lambda) = M_{G_2}(\lambda) - M_{G_2''}(\lambda)$$

$$\therefore \quad M_{G_2}(\lambda) = M_{G_2'}(\lambda) + M_{G_2''}(\lambda)$$

$$= \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+2)^2$$

$$+ \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+3)^2$$

$$= \lambda(\lambda-1)(\lambda-2)\ \ldots\ (\lambda-p+3)\ [\lambda^2 - 2\lambda(p-2) + (p-2)^2 + \lambda-p+3]$$

$$= \lambda(\lambda-1)(\lambda-2)\ \ldots\ (\lambda-p+3)\ [\lambda^2 - \lambda(2p-5) + p^2-5p+7].$$

The equation $\lambda^2 - \lambda(2p-5) + p^2 - 5p + 7 = 0$ has roots given by $\frac{1}{2}(2p - 5 \pm i\sqrt{3})$. We now conclude that, if G is a graph with p vertices and q-2 edges, its chromatic roots are either all integers, or else, the only possible complex chromatic roots are $\frac{1}{2}(2p-5 \pm i\sqrt{3})$.

3.3 <u>Some computational results</u>

As a consequence of direct computation using previously known theorems, we have found the following results.

<u>Result</u> 3.3.1. The chromatic polynomials of graphs with p vertices and 4 edges assume only one of the following three forms.

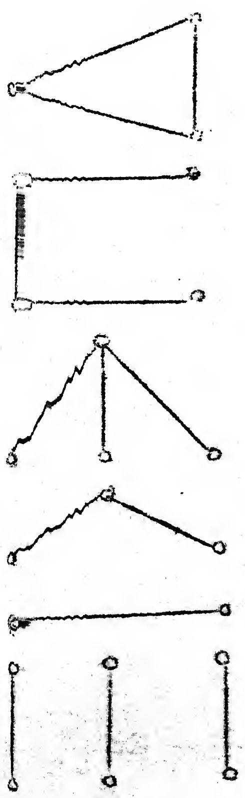
$$\lambda^{p-4}(\lambda-1)^4 \qquad \text{(i)}$$

$$\lambda^{p-3}(\lambda-1)^2(\lambda-2) \qquad \text{(ii)}$$

$$\lambda^{p-3}(\lambda-1)(\lambda^2-3\lambda+3) \qquad \text{(iii)}$$

The equation $(\lambda^2 - 3\lambda + 3) = 0$ has roots $\frac{1}{2}(3 \pm i\sqrt{3})$. We now conclude that the chromatic roots of graphs with p vertices and 4 edges are either all integers, or else, the only possible complex chromatic roots are $\frac{1}{2}(3 \pm i\sqrt{3})$. These complex roots occur for a graph in which the four edges form C_4.

<u>Result</u> 3.3.2. Let q be the number of edges in K_p. There are exactly five graphs with p vertices and q-3 edges. For, let G be a graph with p vertices and q-3 edges. Then on adding three edges suitably to G, it becomes K_p. The three missing edges in G may have one of the following five forms.



In the first two cases,

$$M_G(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+4)(\lambda-p+3)^3 \qquad \text{(i)}$$

In the third case,

$$M_G(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+4)^2(\lambda-p+3)(\lambda-p+2) \qquad \text{(ii)}$$

In the fourth case,

$$M_G(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+4)(\lambda-p+3)\left[\lambda^2 - \lambda(2p-6) + (p^2 - 6p + 10)\right] \qquad \text{(iii)}$$

In the last case,

$$M_G(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-p+4)\left[\lambda^3 - \lambda^2(3p-9) + \lambda(3p^2 - 18p + 29) - (p^3 - 9p^2 + 29p - 34)\right] \qquad \text{(iv)}$$

From (iii), we find that there are two complex chromatic roots given by $\frac{1}{2}(2p - 6 \pm 2i) = p - 3 \pm i$.

In (iv), suppose we consider

$$\lambda^3 - \lambda^2(3p-9) + \lambda(3p^2-18p+29) - (p^3-9p^2+29p-34) = 0$$

This can be written as

$[\lambda - (p-3)]^3 + 2\lambda - 2p + 7 = 0$. This equation has exactly one irrational root between $p-4$ and $p-3$.

Thus, if G is a graph with p vertices and q-3 edges, it is possible that G has complex chromatic roots $p - 3 \pm i$ or an irrational root between p-4 and p-3.

3.4. Some unsolved problems and conjectures

We end up the discussion on chromatic polynomials by mentioning a few unsolved problems and conjectures. Some of these problems and conjectures can be found in [17], [25].

Problem 3.4.1. What is a necessary and sufficient condition for two graphs to have the same chromatic polynomial?

Problem 3.4.2. What are the sufficient conditions that a given polynomial may be the chromatic polynomial of some graph?

Conjecture 3.4.3. It is observed that the coefficients of chromatic polynomials first increase in absolute magnitude and then decrease; two successive coefficients may be equal, but never a coefficient is flanked by larger coefficients. It may be conjectured that this is always true.

Conjecture 3.4.4. All complex chromatic roots have non-negative real parts.

<u>Conjecture</u> 3.4.5. Let G be a graph with p vertices. Let $r = a + ib$ be a root of $M_G(\lambda) = 0$. The maximum possible value of 'a' is $\frac{1}{2}(2p - 5)$ and this occurs when G is the graph obtained from K_p by removing two disjoint edges.

<u>Conjecture</u> 3.4.6. For any fixed number of vertices $p > 6$, there exist graphs with chromatic roots $\frac{1}{2}(r \pm i\sqrt{3})$ where r is an odd integer and $r \leq 2p - 5$ and with chromatic roots $\frac{1}{2} r \pm i$ where r is an even integer and $r < 2p - 5$.

<u>Conjecture</u> 3.4.7. From section 3.1, it is obvious that two non-isomorphic graphs may have the same chromatic polynomial; however, if two non-isomorphic graphs have the same chromatic polynomial, then their complements certainly have different chromatic polynomials. This leads one to the conjecture that this is always the case.

<u>Conjecture</u> 3.4.8. A graph and its complement do not have the same chromatic polynomial unless they are isomorphic.

Bibliography

The starred entries in the bibliography denote books.

1. R.A. Bari, Chromatically equivalent graphs, Graphs and Combinatorics, Lecture Notes in Mathematics 406 (ed. R.A. Bari and F. Harary), Springer-Verlag, Berlin, Heidelberg, New York, 1974, pp. 186-200.

2. R.A. Bari, Chromatic polynomials and the internal and external activities of Tutte, in Graph Theory and Related Topics (ed. J.A. Bondy and U.S.R. Murthy), Academic Press, New York, 1979.

*3. M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs, Prindle, Weber and Schmidt, Boston, Mass., 1979.

*4. C. Berge, Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.

5. G. Berman and W.T. Tutte, The golden root of a chromatic polynomial, J. Combinatorial Theory 6 (1969), 301-302.

6. N.L. Biggs and G.H.J. Meredith, Approximations for chromatic polynomials, J. Combinatorial Theory Ser. B 20 (1976), 5-19.

7. G.D. Birkhoff, The reducibility of maps, Amer. J. Math. 35 (1912), 115-128.

8. G.D. Birkhoff, A determinant formula for the number of ways of coloring a map, Ann. of Math. (2) 14 (1912-3), 42-46.

9. G.D. Birkhoff, On the number of ways of coloring a map, Proc. Edinburg Math. Soc. (2) 2 (1930), 83-91.

10. G.D. Birkhoff and D.C. Lewis, Chromatic polynomials, Trans. Amer. Math. Soc. 60 (1946), 355-451.

*11. J.A. Bondy and U.S.R. Murthy, Graph Theory with Applications, American Elsevier, New York, and MacMillan, London, 1976.

12. R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941), 194-197.

13. V. Chvatal, A note on coefficients of chromatic polynomials, J. Combinatorial Theory 9 (1970), 95-96.

14. B. Daniel, New methods to color the vertices of a graph, Commun. ACM 22 (1979), 251-256.

15. B. Eisenberg, Generalized lower bounds for the absolute values of the coefficients of chromatic polynomials, Recent Trends in Graph Theory, Lecture Notes in Mathematic 186 (ed. M. Capobianco, J.B. Frechen and M. Krolik), Springer-Verlag, Berlin, Heidelberg, New York, 1971, pp. 85-94.

16. E.J. Farrell, On a general class of graph polynomials, J. Combinatorial Theory Ser. B 26 (1979), 111-122.

17. E.J. Farrell, Chromatic roots - Some observations and conjectures, Discrete Math. 29 (1980), 161-167.

18. D.W. Hall, J.W. Siry and B.R. Vanderslice, The Chromatic polynomial of the truncated icosahedron, Proc. Amer. Math. Soc. 16 (1965), 620-628.

*19. F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

20. S.G. Hoggar, Chromatic polynomials and logarithmic concavity, J. Combinatorial Theory Ser. B 16 (1974), 248-254.

21. R.R. Korfhage, σ-Polynomials and graph coloring, J. Combinatorial Theory Ser. B 24 (1978), 137-153.

22. G.H.J. Meredith, Coefficients of chromatic polynomials, J. Combinatorial Theory Ser. B 13 (1972), 14-17.

23. J.F. Nagle, A new subgraph expansion for obtaining colouring polynomials for graphs, J. Combinatorial Theory 10 (1971), 42-59.

24. E.A. Nordhaus and J.W. Gaddum, On complementary graphs, Amer. Math. Monthly 63 (1956), 175-77.

25. R.C. Read, An introduction to chromatic polynomials, J. Combinatorial Theory 4 (1968), 52-71.

26. R.C. Read, Some applications of computers in graph theory, in Selected Topics in Graph Theory (ed. L.W. Beineke and R.J. Wilson), Academic Press, London, 1978.

27. M.R. Sridharan, Graph theory, Q.I.P. Lecture Notes on Linear Graph Theory and its applications (ed. M.R. Sridharan), 1977.

28. W. T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1953), 80-91.

29. W.T. Tutte, On dichromatic polynomials, J. Combinatorial Theory 2 (1967), 301-320.

30. W.T. Tutte, On chromatic polynomials and the golden ratio, J. Combinatorial Theory 9 (1970), 289-296.

31. W.T. Tutte, Codichromatic graphs, J. Combinatorial Theory Ser. B 16 (1974), 168-174.

32. W.T. Tutte, The graph of the chromial of a graph, Combinatorial Mathematics III, Lecture Notes in Mathematics 452 (ed. Anne Penfold Street and W.D. Wallis), Springer-Verlag, Berlin, Heidelberg, New York, 1975, pp. 55-61.

33. H. Whitney, A logical expansion in Mathematics, Bull. Amer. Math. Soc. 38 (1932), 572-579.

34. H. Whitney, The coloring of graphs, Ann. of Math. (2) 33 (1932), 688-718.

35. H.S. Wilf, The Mobius function in combinatorial analysis and chromatic graph theory, in Proof Techniques in Graph Theory (ed. F. Harary), Academic Press, New York, 1969.

A 62321

Date Slip A 62321

This book is to be returned on the date last stamped.

....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....
....	....

CD 6.72.9

MATH-1990-M-SWA-CHR